Final Exam of Analytic Number Theory in 2023 Fall

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Collating: Mathzwj

1. Let q be a positive integer. Write down the definition of $L(s,\chi)$ for $s>1$ and a Dirichlet character to the modulus q. Prove that

$$\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\log L(s,\chi)=\sum_{p}\sum_{m:p^m\equiv a(\mathrm{mod}\, q)} m^{-1}p^{-ms}.$$

2. For $s>1$, show that $\zeta^2(s)=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}$, where $\tau(n)=\sum_{d|n}1$.

3. Show shat there are infinitely many $n\in\mathbb{N}$ such that $\mu(n)=\mu(n+1)$, where $\mu(n)$ is Möbius function.

4. Prove that $\sum_{n\le x}\Lambda^2(n)\sim x\log x$ as $x\to+\infty$.

5. Show that $\prod_{p\le z}\left(1-\frac{1}{p}\right)^{-1}>>\log z$.

6. Suppose $P>1$, $Q\ge 2P$. For integers a,q such that $1\le a\le q\le P$, let $\mathfrak{M}(a,q)$ define the interval $\{\alpha:|\alpha-a/q|\le 1/qQ\}$. Prove that if $a/q\ne a'/q'$, then $\mathfrak{M}(a,q)$ and $\mathfrak{M}(a',q')$ are disjoint.

7. Prove that $\frac{\sigma(n)}{n}=\frac{\pi^2}{6}\sum_{q=1}^{\infty}\frac{c_q(n)}{q^2}$ for positive integers n, where $\sigma(n)=\sum_{d|n}d$ and $c_q(n)=\sum_{1\le a\le q,(a,q)=1}e\left(\frac{an}{q}\right)$.

8. Write down three results in analytic number theory proved after 2000.

9. Write down five unsolved problems in analytic number theory.

10. Write down your plan to study analytic number theory in the future.

11. Write down your suggestions for the course.